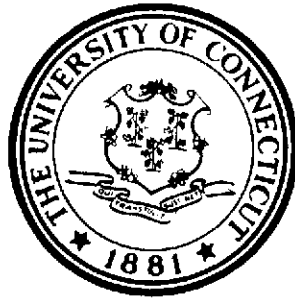


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(NASA-CR-133429) A REDUCED ADAPTIVE
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A REDUCED ADAPTIVE OBSERVER
FOR MULTIVARIABLE SYSTEMS

by

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1. Introduction

The adaptive observer concept is a scheme for determining the state of a system possessing unknown parameters when only the system input, output, and form are known. The first reported adaptive observer, for single-input single-output time invariant linear systems, appeared in [1] and [2]. A modification of this observer to simplify the adaptive dynamics was subsequently reported [3]. Both these schemes exhibit the desirable properties that the eigenvalues of the observer matrix may be freely chosen (an important capability for systems with measurements corrupted by noise), that the simple Liapunov adaptive algorithm is implemented entirely on line during system operation, that no derivative networks are required in the adaptive algorithm, and that both the state of the system under observation and the unknown parameters of that system are progressively determined regardless of the magnitude of parameter ignorance.

In [4] the single-input single-output adaptive observer was extended to cyclic multivariable systems by introducing a suitable transformation that converts the system to a single-output system. Consequently the multivariable adaptive observer in this scheme is of the same order as the system regardless of the number of system outputs available, and the number of adaptive gains needed to implement the observer algorithm equals at least the sum of the system order and the number of input parameters being adapted.

In this paper an adaptive observer for multivariable systems is reported for which the dynamic order of the observer is reduced, subject to mild restrictions given in Theorem 1, to $n-p+1$ where n is the order of the system being observed and p is the number of independent output measurements. The observer structure which is developed here depends directly upon the

multivariable structure of the system rather than a transformation to a single-output system. The number of adaptive gains is at most the sum of the order of the system and the number of input parameters being adapted. Moreover, for the relatively frequent specific cases for which the number of required adaptive gains is less than the sum of system order and input parameters, the number of these gains is easily determined by inspection of the system structure. This adaptive observer possesses all the properties ascribed earlier to the single-input single-output adaptive observer. Like the other adaptive observers mentioned, some restriction is required of the allowable system command input to guarantee convergence of the adaptive algorithm, but the restriction is more lenient than that required by ^{the} full-order multivariable observer in [4]. Finally, this reduced observer is not restricted to cyclic systems as is [4].

2. The Problem To Be Solved

An observable and controllable linear time-invariant dynamical system described by

$$\begin{aligned}\dot{\bar{x}} &= A\bar{x} + Br & \bar{x}(t_0) &= \bar{x}^0 \\ y &= C\bar{x}\end{aligned}\tag{1}$$

is considered, where $\bar{x}(t) \in \mathbb{E}^n$ is the state of the system, $r(t) \in \mathbb{E}^m$ is the command input and $y(t) \in \mathbb{E}^p$ is the output. For purposes of this paper, (1) is multivariable with $n > p \geq 1$ and $m \geq 1$, the pair (C, A) is completely observable, and the pair (A, B) is completely controllable. A and B are appropriately - dimensioned matrices having parameters of unknown value. C is a known matrix of dimension $p \times n$.

The problem is to determine the state \bar{x} of (1) using only the input r , the output y , and the structure, but not the values, of matrices A and B .

This is to be accomplished by a generating process which duplicates as little as possible the state information available in the output; thus the generating process is said to be a reduced adaptive observer.

The reduced adaptive observer is of the form

$$\begin{aligned}\dot{\xi} &= F\xi + Gy + Dr + Hu \\ \xi(t_0) &= \xi^0\end{aligned}\tag{2}$$

where $\xi \in \mathbb{R}^{n-p+1}$ is the estimate of the missing state information in the output of (1). The matrices G and D and the vector u are to be adaptively manipulated so as to guarantee that ξ asymptotically equals a transformation of the unknown state variables in (1). F may have arbitrary distinct eigenvalues.

The state \bar{x} can be ultimately constructed once the transformation has been identified. Figure 1 illustrates the situation.

3. The Strategy of the Solution

The transformation T , indicated by T^{-1} in Figure 1, allows the system (1) to be assumed to be in a form suitable for constructing an adaptive law based upon Liapunov synthesis techniques.

The strategy for solving the problem posed in Section 2 is to first determine the effects of parameter uncertainty in the system upon the accuracy of the observer estimate of the system state. In Section 4 an error vector is defined as a comparison between the transformed system state and the observer estimate; subsequently an error equation is derived reflecting the influences of parameter uncertainty in the system. Theorem 1 of Section 5 defines sufficient conditions under which (1) may be transformed into a form suitable for a Liapunov synthesis technique. It is seen in this section that with this form the error equation may be considerably simplified. In Section 6 the Liapunov adaptive synthesis technique is used to derive an adaptive law. The

essence of this method is to define the adaptable parameters in such a way as to insure, by means of a Liapunov function, that the error equation is asymptotically stable. Due to the fact that the resulting Liapunov function chosen here (as may be seen by equation (18) and (20)) is defined on a non-compact manifold, Theorem 2 gives sufficient restriction upon the system input to insure that the error equation is asymptotically stable on the compact manifold. Thus an estimate of the system state, which asymptotically converges to the true system state, may be obtained by inverting the original transformation of the system as indicated in Section 7.

As an illustration of the technique of this paper, an example is given in Section 8 and a computer simulation of this example appears in Section 9.

It is propitious to collect here certain definitions which allow brevity in the remaining sections of this paper. The motivations for these definitions will be discussed in the appropriate locations.

Definition 1

$\mathcal{J}_{n,p}$ hereafter refers to the collection of all non-singular square matrices T of dimension $n \times n$ having the following properties

T may be partitioned as

$$T = \begin{array}{cc} & \begin{array}{cc} (p-1) \times (p-1) & (p-1) \times (n-p+1) \end{array} \\ \begin{bmatrix} T_{11} & \vdots & T_{12} \\ \hline T_{21} & \vdots & T_{22} \end{bmatrix} \\ & \begin{array}{cc} (n-p+1) \times (p-1) & (n-p+1) \times (n-p+1) \end{array} \end{array}$$

wherein a) $T_{12} = 0$;

b) each element in the uppermost row of T_{21} is independent of any system parameter.

and c) the uppermost row of T_{22} is $[C \ 0 \ 0 \ \dots \ 0]$ with $C \neq 0$.

When there is no possibility of confusion, $\mathcal{J}_{n,p}$ will be referred to as \mathcal{J} .

Definition 2

The "adaptive canonical form" refers to all matrices \tilde{A} of dimension $n \times n$ having the following properties:

in the partition

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}$$

a) \tilde{A}_{22} has the form

$$\begin{bmatrix} \alpha_1 & 1 & 1 & 1 & \dots & 1 \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & \Lambda_{n-p} & & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix}_{(n-p+1) \times (n-p+1)}$$

where Λ_{n-p} is a diagonal matrix with distinct eigenvalues of dimension $n-p$ and α_1 any real number,

and b) \tilde{A}_{21} has no more than $n-1$ non-zero elements.

4. The Error Equation

The development of the error equation is somewhat similar to that in [5] for systems with known parameters.

Without restriction if C is known, it may be assumed in (1) that $C = [I_p \ 0]$ where I_p is a $p \times p$ identity matrix. For example, the transformation $\bar{x} = \begin{bmatrix} C \\ H \end{bmatrix}^{-1} x$ transforms (1) into this form for any C, where H is selected to make the transformation matrix non-singular. Then in partitioned form (1) is written as

$$\begin{aligned} \dot{\bar{y}} &= A_{11}\bar{y} + A_{12}\dot{w} + B_1 r \\ \dot{w} &= A_{21}\bar{y} + A_{22}w + B_2 r \end{aligned} \quad (3)$$

$$y = \begin{bmatrix} \bar{y} \\ w_1 \end{bmatrix} \quad x = \begin{bmatrix} \bar{y} \\ w \end{bmatrix}$$

where $\bar{y} \in \mathcal{E}^{p-1}$ and $w \in \mathcal{E}^{n-p+1}$. The dimensions of \bar{y} and w indicate the dimensions of the partitions in (3). Since only w is to be estimated by the adaptive observer, the dimension of the vector w is chosen as small as possible while still retaining an element of the output, which is essential for implementation of the adaptive law.

The adaptive observer is initially described by

$$\dot{\xi} = F\xi + (FK + G - KM)\bar{y} - K\bar{y} + (D - KB_1)r + Hu \quad (4)$$

in which $\xi \in \mathcal{E}^{n-p+1}$. If at this point (4) is taken as a hypothesis for a generator of w , it will be shown that the error between w and a function of ξ can be made to vanish by adaptively adjusting G , D , and K . It will be subsequently shown that a suitable transformation of (3) allows (4) to be rendered unto (2).

Let an estimate of w be $\xi + K\bar{y}$. Then defining, as in [5], the error

$$e = \xi + K\bar{y} - w \quad (5)$$

on the reduced space \mathcal{E}^{n-p+1} , it follows that

$$\begin{aligned}\dot{\bar{e}} &= \dot{\xi} + \dot{K}\bar{y} + \dot{K}\bar{y} - \dot{w} \\ &= F(\xi + K\bar{y} - w) + (F - A_{22})w + (G - A_{21})\bar{y} \\ &\quad + K(\dot{\bar{y}} - M\bar{y}) + (D - B_2 - KB_1)r + Hu\end{aligned}$$

Defining $M = \tilde{M} - A_{11}$ so that $\dot{\bar{y}} - M\bar{y} = \dot{\bar{y}} - A_{11}\bar{y} - \tilde{M}\bar{y} = A_{12}w + B_1r - \tilde{M}\bar{y}$, then

$$\begin{aligned}\dot{\bar{e}} &= Fe + (F - A_{22} + KA_{12})w + (G - A_{21} - KM)\bar{y} \\ &\quad + (D - B_2)r + Hu\end{aligned}\tag{6}$$

in which it is seen that the reduced error depends upon both the measurable vector \bar{y} and the unmeasurable (save the first element) vector w . It is impossible in a manner similar to the Luenberger observer [6,7] to define $F = A_{22} - KA_{12}$, $M = A_{11}$, $D = B_2$, $G = A_{21}$, and $H = 0$ to eliminate these dependences from (6), since A and B are here unknown.

Rather, it is desired to adaptively adjust the triple (G, D, K) so that the coefficients of w , \bar{y} , and r in (6) eventually vanish. Then if F is chosen with eigenvalues all with negative real parts and if $u \rightarrow 0$, the reduced error e vanishes.

5. The Transformation

If it is possible to show with respect to (3) that a suitable transformation matrix T exists so that $T_{11} = CT$ and that $\tilde{A} = T^{-1}AT$ is in adaptive canonical form with the partition element $\tilde{A}_{22} = T_{22}^{-1}(A_{22} - T_{21}A_{12})T_{22}$ having arbitrary specified eigenvalues, then setting $F = \tilde{A}_{22}$ in the equation (analogous to (6))

$$\begin{aligned}\dot{\bar{e}} &= Fe + (F - \tilde{A}_{22} + K\tilde{A}_{12})w + (G - \tilde{A}_{21} - KM)\bar{y} \\ &\quad + (D - \tilde{B}_2)r + Hu\end{aligned}\tag{7}$$

permits defining $K = 0$. Doing this is advantageous since the influence of w in (7) is eliminated, the necessity of adapting K is removed, and (since \tilde{M} is related to \tilde{A}_{11}) the influence of unknown elements of $\tilde{A}_{11} = T_{11}^{-1} (A_{11}T_{11} + A_{12}T_{21})$ is diminished.

As will be seen, under some restrictions on A a transformation T can be found that satisfies the preceding requirement and the additional requirement that $T_{11} = CT$ be independent of parameters of A . By virtue of this latter requirement the outputs can be treated as transformed state variables without specifically identifying T .

For suitable definition of u , the transformation which satisfies these requirements is a member of the collection \mathcal{J} and the transformed matrix $\tilde{A} = T^{-1}AT$ is in adaptive canonical form. Theorem 1 gives sufficient conditions on A for the existence of such a $T \in \mathcal{J}$.

In the following theorem, let the symbol $\mathcal{R}[\chi]$ denote the range of χ , let $Q \equiv [C \ 0 \ 0 \ \dots \ 0]$, $C \neq 0$ be a row vector of dimension $1 \times (n-p+1)$, and let \tilde{A}_{22} denote the $(n-p+1) \times (n-p+1)$ partition of the adaptive canonical form.

Theorem 1

Let the pair (A_{12}, A_{22}) of the matrix A be completely observable. Then there exists a $T \in \mathcal{J}$ that transforms A into the adaptive canonical form in which $n-p$ eigenvalues of \tilde{A}_{22} may be almost arbitrarily chosen.

If in addition, $\mathcal{R}[Q^T] \subset \mathcal{R}[A_{12}^T]$, then the $n-p$ eigenvalues of \tilde{A}_{22} may be arbitrarily chosen.

Proof. The proof is in two parts: to show that a $T \in \mathcal{J}$ exists that puts \tilde{A}_{22} into adaptive canonical form with the desired properties, and that \tilde{A}_{21} also satisfies the requirements of the adaptive canonical form.

Suppose that (A_{22}, A_{12}) of the matrix A is completely observable. According to the definition of \mathcal{J} , the $(n-p+1) \times (n-p+1)$ partition T_{22} of $T \in \mathcal{J}$ is arbitrary except for the uppermost row which is $Q \equiv [C \ 0 \ 0 \ \dots \ 0]$, $C \neq 0$. Since $\tilde{A}_{22} = T_{22}^{-1} (A_{22} - T_{21}A_{12})T_{22}$, where T_{21} is the $(n-p+1) \times (p-1)$ partition of $T \in \mathcal{J}$, it must be shown that \tilde{A}_{22} is of the form required by definition of the adaptive canonical form, and that by choice T_{21} the $n-p$ eigenvalues can be freely chosen. It has been shown [3] that there exists a matrix T_{22} of the required form which transforms a cyclic matrix P into $\tilde{A}_{22} + L$, where L is a matrix having only the leftmost column non-zero, if and only if (Q, P) is completely observable. In the present context, $P \equiv A_{22} - T_{21}A_{12}$. Thus if by choice of T_{21} , P can have $n-p$ eigenvalues equal to the desired eigenvalues of \tilde{A}_{22} and if (Q, P) is completely observable for this choice of T_{21} , then $L = 0$ (except perhaps for the element in the upper left corner, which is irrelevant by definition of the adaptive canonical form).

Suppose first that $\mathcal{R}[Q^T] \subset \mathcal{R}[A_{12}^T]$. Then for any choice of T_{21} the pair $(Q, A_{22} - T_{21}A_{12})$ is completely observable and at least $n-p$ eigenvalues of $A_{22} - T_{21}A_{12}$ can be arbitrarily chosen [8]. Therefore $\tilde{A}_{22} = T_{22}^{-1} (A_{22} - T_{21}A_{12}) T_{22}$ is in adaptive canonical form with arbitrary eigenvalues for some choice of T_{21} and T_{22} of $T \in \mathcal{J}$.

Suppose now that $\mathcal{R}[Q^T] \not\subset \mathcal{R}[A_{12}^T]$. Since the pair (A_{12}, A_{22}) is completely observable, at least $n-p$ eigenvalues of $P = A_{22} - T_{21}A_{12}$ can be arbitrarily chosen but (Q, P) may not be observable. A trivial extension of Theorem 4 of [10] says that the set $K \equiv \{T_{21} | (A_{22} - T_{21}A_{12}, Q) \text{ not observable}\}$ is either an empty set or a hypersurface in the parameter space of T_{21} when the pair (A_{22}, A_{12}) is completely observable. Consequently \tilde{A}_{22} is in adaptive canonical form with almost arbitrary eigenvalues for some choice of T_{22} and T_{21} of $T \in \mathcal{J}$, since the choices of T_{21} is limited to those $T_{21} \notin K$. Thus the first part of the theorem is proved.

Now it is shown that \tilde{A}_{21} has no more than $n-1$ non-zero elements with the appropriate choice of $T \in \mathcal{J}$. In the $(n-p+1) \times (p-1)$ partition T_{21} of $T \in \mathcal{J}$ there are $(n-p+1)(p-1) - (p-1) = (n-p)(p-1)$ parameter-dependent elements. At most $n-p$ of these elements are needed to specify the $n-p$ eigenvalues of $A_{22} - T_{21}A_{12}$. Therefore, at least $(n-p)(p-1) - (n-p) = (n-p)(p-2)$ parameter-dependent elements of T_{21} are unspecified. Each unspecified element may be specified so as to make an element of $\tilde{A}_{21} = T_{22}^{-1} (A_{21}T_{11} + A_{22}T_{21} - T_{21}T_{11}^{-1}A_{11}T_{11} - T_{21}A_{12}T_{21})$ zero. Since there are at most $(n-p+1)(p-1)$ non-zero elements in A_{12} , eliminating

$(n-p)(p-2)$ of them leaves at most $(n-p+1)(p-1) - (n-p)(p-2) = n-1$ non-zero elements in A_{21} . Thus the theorem is proved.

Corollary

If in addition to the requirements of the theorem the pair (Q, A_{22}) is completely observable where $Q = [C \ 0 \ 0 \ \dots \ 0]$, $C \neq 0$, then the uppermost row of the partition T_{21} of $T \in \mathbb{F}$ may be chosen as zero.

Proof

The proof of the theorem requires that $(Q, A_{22} - T_{21}A_{12})$ be completely observable for some choice of T_{21} . If (Q, A_{22}) is observable, then $(Q, A_{22} - T_{21}A_{12})$ is completely observable by the trivial choice $T_{21} = 0$. However, since $n-r$ eigenvalues of $A_{22} - T_{21}A_{12}$ are to be arbitrarily chosen by choice of T_{21} and at least $(n-p)(p-2)$ elements of \tilde{A}_{21} are to be chosen zero by choice of T_{21} , it generally requires all but $p-1$ non-zero elements of T_{21} . Generally these elements must be parameter-dependent; thus only the $p-1$ parameter-independent elements appearing in the uppermost row of T_{21} may be zero.

6. The Adaptive Law

It is assumed that (3) satisfies the conditions of Theorem 1 and consequently may be written as

$$\begin{aligned}\dot{\bar{y}} &= \tilde{A}_{11}\bar{y} + \tilde{A}_{12}w + \tilde{B}_1r \\ \dot{w} &= \tilde{A}_{21}\bar{y} + \tilde{A}_{22}w + \tilde{B}_2r \\ y &= \begin{bmatrix} \bar{y} \\ w \end{bmatrix}\end{aligned}\tag{8}$$

where \tilde{A} is of adaptive canonical form and the scalar w is a linear combination of w_1 and elements of \bar{y} . The scalar w is constructed externally to the system in accordance with the upper row of T_{21} so that the transformed system output matrix is in the form assumed in (3). According to the corollary, $w = w_1$ if (Q, A_{22}) is completely observable.

F in (2) is taken as

$$F = \begin{bmatrix} -\lambda_1 & 1 & 1 & 1 & \dots & 1 \\ 0 & & & & & \\ 0 & & & & & \\ 0 & & \Lambda_{n-p} & & & \\ \vdots & & & & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix} \quad (n-p+1) \times (n-p+1) \quad (9)$$

for λ_1 any real number distinct from the distinct eigenvalues of the diagonal matrix Λ_{n-p} . Let $h_1 u_1 = (\alpha + \lambda_1 - g_1) w_1 \equiv \phi_1 w_1$ and (7) may be written

$$\dot{e} = Fe + \Phi \ddot{y} + \Psi \dot{r} + Hu \quad (10)$$

$$e(t_0) = e^0$$

where $\Phi = G - \tilde{A}_{21}$ and $\Psi = D - \tilde{B}_2$. The other elements of u will be defined later.

The adaptive law for Φ and Ψ in (10) must be defined in terms of only those variables which are available for measurement. Consequently (10) will be manipulated in a way to obtain a scalar equation, equivalent to (10), for which such an adaptive law can be formulated.

Let the $(n-p+1) \times (p-1)$ matrix V be defined as

$$V = \begin{bmatrix} \ddot{y}^T \\ \vdots \\ (pI - \Lambda)^{-1} h_1 \ddot{y}^T \end{bmatrix} \quad (11)$$

in which p means " $\frac{d}{dt}$ " and $h_1^T = [1 \ 1 \ 1 \ \dots \ 1]$ of appropriate dimension.

Clearly the $(n-p) \times (p-1)$ submatrix $(pI - \Lambda_{n-p})^{-1} h_1 \ddot{y}^T$ is composed of filtered output variables.

In a similar manner, let the $(n-p+1) \times m$ matrix S be defined as

$$S = \begin{bmatrix} r^T \\ \hline (pI - \Lambda)^{-1} h_2^T r^T \end{bmatrix} \quad (12)$$

in which $h_2^T = [1 \ 1 \ 1 \dots 1]$ of appropriate dimension. It will later be shown that the adaptive law requires at most $n-1$ elements of V .

Consider now the lowermost $n-p$ scalar equations of (10). The i^{th} equation, $2 \leq i \leq n-p+1$, is

$$\dot{e}_i = -\lambda_i e_i + \sum_{j=1}^{p-1} \phi_{ij} y_j + \sum_{j=1}^m \psi_{ij} r_j + h_i u_i \quad (13)$$

If in (13) $h_i u_i$, $2 \leq i \leq n-p+1$, is defined as

$$\begin{aligned} h_1 u_1 &= \phi_1 w_1 \\ h_i u_i &= \sum_{j=1}^{p-1} \dot{\phi}_{ij} v_{ij} + \sum_{j=1}^m \dot{\psi}_{ij} s_{ij} \end{aligned} \quad (14)$$

$$2 \leq i \leq n-p+1$$

then (13) is a separable differential equation for each i . To show this, the identities for each i

$$\sum_{j=1}^{p-1} \dot{\phi}_{ij} v_{ij} = \frac{d}{dt} \left[\sum_{j=1}^{p-1} \phi_{ij} v_{ij} \right] - \sum_{j=1}^{p-1} \phi_{ij} \dot{v}_{ij}$$

$$\sum_{j=1}^m \dot{\psi}_{ij} s_{ij} = \frac{d}{dt} \left[\sum_{j=1}^m \psi_{ij} s_{ij} \right] - \sum_{j=1}^m \psi_{ij} \dot{s}_{ij}$$

are needed. Using them, (13) becomes

$$\begin{aligned} \dot{e}_i &= -\lambda_i e_i + \sum_{j=1}^{p-1} \phi_{ij} y_j + \sum_{j=1}^m \psi_{ij} r_j \\ &\quad - \sum_{j=1}^{p-1} \phi_{ij} \dot{v}_{ij} - \sum_{j=1}^m \psi_{ij} \dot{s}_{ij} - \\ &\quad + \frac{d}{dt} \left[\sum_{j=1}^{p-1} \phi_{ij} v_{ij} + \sum_{j=1}^m \psi_{ij} s_{ij} \right] \end{aligned}$$

Substituting (11) and (12) into the above yields

$$\frac{d}{dt} \left[e_i - \sum_{j=1}^{p-1} \phi_{ij} v_{ij} - \sum_{j=1}^m \psi_{ij} s_{ij} \right] = -\lambda_i \left[e_i - \sum_{j=1}^{p-1} \phi_{ij} v_{ij} - \sum_{j=1}^m \psi_{ij} s_{ij} \right] \quad (15)$$

(15) is integrated to yield

$$e_i = \sum_{j=1}^{p-1} \phi_{ij} v_{ij} + \sum_{j=1}^m \psi_{ij} s_{ij} + \theta_i \exp[-\lambda_i t] \quad (16)$$

where

$$\theta_i = e_i^0 - \sum_{j=1}^{p-1} \phi_{ij}(t) v_{ij}(t) - \sum_{j=1}^m \psi_{ij}(t) s_{ij}(t)$$

at $t = t_0$.

Equation (16) is applied to the first equation of (10) giving

$$\dot{e}_1 = -\lambda_1 e_1 + \text{tr } \Phi^T V + \text{tr } \Psi^T S + \phi_1 y_p + \sum_{i=2}^{n-p+1} \theta_i \exp[-\lambda_i t] \quad (17)$$

It is thus seen that (10) and (17) are equivalent, with definition of u in (14), but with the difference that (17) is a scalar equation. The adaptive law, dependent upon measurable variables only, may now be formulated.

A Liapunov function candidate is selected as

$$V = e_1^2 + \delta_1^2 \phi_1^2 + \text{tr } (\Delta \otimes \Phi^T \Phi) + \text{tr } (\Gamma \otimes \Phi^T \Phi) \quad (18)$$

in which Δ and Γ are matrices having no non-positive element and the symbol \odot represents element-by-element multiplication of matrices. The time derivative of (18) along the trajectory described by (17) is

$$\begin{aligned} 2\dot{V} = & -\lambda_1 e_1^2 + (\delta_1^2 \dot{\phi}_1 + y_p e_1) \phi_1 \\ & + \text{tr } \phi^T (\Delta \odot \dot{\phi} + V e_1) + \text{tr } \psi^T (\Gamma \odot \dot{\psi} + S e_1) \\ & + \sum_{i=2}^{n-p+1} \theta_i \exp[-\lambda_i t] e_1 \end{aligned} \quad (19)$$

Then (19) can be made

$$2\dot{V} = -\lambda_1 e_1^2 + \sum_{i=2}^{n-p+1} \theta_i \exp[-\lambda_i t] e_1 \quad (20)$$

whenever $\dot{\phi}_1$, $\dot{\phi}$, and $\dot{\psi}$ are defined as

$$\begin{aligned} \delta_1^2 \dot{\phi}_1 &= -y_p e_1 \\ \Delta \odot \dot{\phi} &= -V e_1 \\ \Gamma \odot \dot{\psi} &= -S e_1 \end{aligned} \quad (21)$$

Equations (21) may be also written in scalar form as

$$\begin{aligned} \dot{\phi}_1 &= -\frac{1}{\delta_1^2} y_p e_1 = -\dot{g}_1 \\ \dot{\phi}_{ij} &= -\frac{1}{\delta_{ij}^2} v_{ij} e_1 = -\dot{g}_{ij} \\ \dot{\psi}_{ij} &= -\frac{1}{\gamma_{ij}^2} s_{ij} e_1 = -\dot{d}_{ij} \end{aligned} \quad (21a)$$

for each i, j in their proper domains. Equations (21) or (21a) are the adaptive laws sought.

V is eventually negative definite whenever all the eigenvalues $-\lambda_1, -\lambda_2, \dots, -\lambda_{n-p+1}$ have negative real parts since then the initial condition disturbances θ_i decay exponentially. Consequently e_1 is asymptotically stable in the sense of Liapunov.

It is desired that $\lim_{t \rightarrow \infty} e(t) = 0$ in order for the adaptive observer to generate the system state. If some restriction on the input vector r is imposed, it can be shown that $e_1 \rightarrow 0$ implies $e \rightarrow 0$.

To see this, consider the limiting value of (17) which is

$$0 = \text{tr } \Phi^T V + \text{tr } \Psi^T S + \phi_1 y_p \quad (22)$$

If by suitably restricting r , or equivalently V , S , and y_p , so that (22) implies in the limit that $\Phi^T = 0$, $\Psi^T = 0$, and $\phi_1 = 0$, then (10) is

$$\dot{e} = Fe$$

implying $e \rightarrow 0$ since F is an asymptotically stable matrix. The above equation follows from (10) since u_i , $2 \leq i \leq n-p+1$, is zero in the limit as evident from (14) and (21).

The following theorem defines the restriction on r guaranteeing $\Phi=0$, $\Psi=0$, and $\phi_1=0$ for $e_1=0$ when the steady state r is periodic.

Theorem 2.

Let q be the number of adaptive parameters in the observer (2), let the observer matrix F have eigenvalues all with negative real parts, and let the system (3) be completely controllable through each column vector in the input matrix B . If the collection of inputs $\{r_1, r_2, \dots, r_m\}$ possesses no fewer than $\lceil (q+1)/2 \rceil$ distinct steady-state frequencies, then (2) generates the system state.

Proof.

The proof is by induction. It is shown [2,3] that the theorem holds for $m=1$. Assuming that the theorem holds for $m=m_1$, it will be shown that it holds for $m=m_1+1$.

Let each y_j , $1 \leq j \leq p$, be related to the inputs $r_1, r_2, \dots, r_{m_1+1}$ by

$$y_j = \sum_{k=1}^{m_1+1} h_{jk}(p) r_k$$

where $p \equiv d/dt$.

Then (22) is

$$0 = \sum_{k=1}^{m_1+1} \left\{ \psi_{1k} + \sum_{j=1}^{p-1} \phi_{ij} h_{jk}(p) + \sum_{i=2}^{n-p+1} \sum_{j=1}^{p-1} \left(\phi_{ij} \frac{h_{jk}(p)}{p+\lambda_i} + \frac{\psi_{ik}}{p+\lambda_i} \right) + \phi_1 h_{pk}(p) \right\} r_k \quad (23)$$

Since, by (20), $e_1 \rightarrow 0$ and, by (21), ϕ_{ij} , ψ_{ij} , and ϕ_1 are constants, (23) may be written

$$H_1(p)r_1 + H_2(p)r_2 + \dots + H_{m_1}(p)r_{m_1} = -H_{m_1+1}(p)r_{m_1+1} \quad (24)$$

where $H_k(p)$ are the terms in brackets in (23) for each k , $1 \leq k \leq m_1+1$. Let the number of distinct adaptive coefficients in the left side of (24) be q_1 and the number of distinct adaptive coefficients in $H_{m_1+1}(p)$ be q_2 . By definition $q=q_1 + q_2$. By assumption $\{r_1, r_2, \dots, r_{m_1}\}$ contains $[(q_1 + 1)/2]$ distinct frequencies and the left side of (24) is zero since $H_1(p) = H_2(p) = \dots = H_{m_1}(p)=0$ and

$$0 = H_{m_1+1}(p)r_{m_1+1}$$

Therefore only the distinct coefficients of $H_{m_1+1}(p)$ are non-zero. By inspection of (23), these are the ψ_{ik} terms which are q_2 in number. Thus by [2,3] if r_{m_1+1} contains at least $[(q)/2]$ distinct frequencies (i.e. distinct from the frequencies of $\{r_1, r_2, \dots, r_{m_1}\}$) then $H_{m_1+1}(p) \neq 0$. Consequently $\{r_1, r_2, \dots, r_{m_1+1}\}$ containing $[(q)/2]$ distinct frequencies implies that $H_1(p) = H_2(p) = \dots = H_{m_1+1}(p)$ which was to be proved.

Theorem 3

Let the conditions on the observer (2) be as stated in Theorem 2, but let there be no requirement upon the column vectors of the input matrix B of the system (3). Then it is sufficient that each input $r_i \in r$ each possess $[(q)/2]$ distinct steady-state frequencies in order for (2) to generate the system state.

Proof:

The proof follows from equation (23). When any $h_{jk}(p)$ is zero or linearly dependent, then the parameters ϕ_{ij} and ψ_{ij} are not fully "coupled" with each of the inputs r_k of equation (23). This in general requires that frequencies must be assigned to each r_k depending upon the degree of freedom in the coefficient of r_k in equation (23). Assuming complete "decoupling" of each ϕ and ψ with respect to each r_k , it is clearly sufficient that each r_k must possess $[(q)/2]$ frequencies from equation (24).

Remark: The sufficient conditions stated in Theorem 3 are noted to be very conservative as a cursory glance at the proof of this theorem reveals. It is suspected by the authors that under the conditions of Theorem 3 the requirement for state generation may be liberalized to allow only the collection of inputs

$\{r_1, r_2, \dots, r_m\}$ possess $[(q)/2]$ steady-state frequencies, as in Theorem 2, but with the additional restriction that the frequencies must be assigned in some way depending upon the controllability structure of system (3).

At the time of this writing however, the above speculation has not been proved.

7. Reconstruction of the System State

The observer (2) generates the state of the transformed system (8). To obtain the state of the system (1) the observer estimate ξ must be transformed by

$$\hat{x} = T \begin{bmatrix} \bar{y} \\ \xi \end{bmatrix}$$

where \hat{x} is the estimate of the system state x . T cannot be immediately written since it contains unknown elements of A ; however, sufficient identification of the system matrix A occurs as a result of the adaptive laws (21) to allow T to be determined. Consequently the time-varying matrix $\hat{T}(G,D)$ may be constructed so that

$$\hat{x} = \hat{T}(G,D) \begin{bmatrix} \bar{y} \\ \xi \end{bmatrix}$$

is the observer estimate of x . Since $\lim_{t \rightarrow \infty} \hat{T}(G,D) = T$ the state x is obtained.

Theorem 4 summarizes the results of this paper.

Theorem 4

The state of system (1) may be adaptively constructed by the observer (2) by employing the adaptive algorithm (21) and the control vector u of (14), both subject to definitions (11) and (12) if

- a) in (1) the partition (A_{12}, A_{22}) is completely observable, and
- b) the number of distinct frequencies in the system command input r is no fewer than $[(q + 1)/2]$ where q is the number of parameters to be adapted. Moreover, the number of parameters to be adapted is not greater than n plus the number of input parameters.

8. Example

A specific example is given here to illustrate the design of a reduced-order adaptive observer.

Suppose the system is represented by

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{x}}_3 \\ \dot{\bar{x}}_4 \end{bmatrix} = \begin{bmatrix} -a_3 & 1 & 0 & 0 \\ -a_2 & 0 & 1 & 0 \\ -a_1 & 0 & 0 & 1 \\ -a_0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b_1 & 0 \\ 0 & 0 \\ 0 & b_2 \end{bmatrix} r \quad (1^*)$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix}$$

with a_0, a_1, a_2, a_3, b_1 , and b_2 unknown constants. (This is, of course, not the most general input matrix.)

It is seen that $(A_{12}, A_{22}) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$ is completely observable.

Therefore there exists a non-singular square transformation $T \in \mathcal{J}$ that puts (1*) into adaptive canonical form. Such a matrix is

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_1 & -\lambda_2^2 & \lambda_2 & 1 \end{bmatrix}$$

Note that the uppermost row of T_{21} is zero since $([1 \ 0], A_{22})$ is a completely observable pair.

Then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -a_3 & 1 & 0 & 0 \\ -a_2 & 0 & 1 & 0 \\ 0 & -\lambda_2^2 & \lambda_2 & 1 \\ \tau & \lambda_2^3 - a_1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b_1 & 0 \\ 0 & 0 \\ b_1 \lambda_2^2 & b_2 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \quad (8^*)$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x$$

where $\tau = a_1 a_3 - a_2 \lambda_2^2 - a_0$

From the form of (3*) it is seen that

$$\begin{aligned} \phi_{21} &= e_{21} - \tau \\ \phi_{22} &= e_{22} + (a_1 - \lambda_2^3) \\ \phi_{21} &= d_{21} - b_1 \lambda_2^2 \\ \phi_{22} &= d_{22} - b_2 \end{aligned} \quad (10^*)$$

is to be adapted. Note that only 4 parameters need to be adapted for which in (1*) there are 6 unknowns.

The adaptive laws are

$$\begin{aligned}\dot{\hat{g}}_{21} &= -\frac{1}{\delta_{21}} v_{21} e_1 \\ \dot{\hat{g}}_{22} &= -\frac{1}{\delta_{22}} v_{22} e_1 \\ \dot{\hat{d}}_{21} &= -\frac{1}{\gamma_{21}} s_{21} e_1 \\ \dot{\hat{d}}_{22} &= -\frac{1}{\gamma_{22}} s_{22} e_1\end{aligned}\tag{21*}$$

in which $e_1 = \xi_1 - y_3$, and the reduced observer is

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 1 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} 0 & -\lambda_2^2 \\ \hat{g}_{21} & \hat{g}_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \hat{d}_{21} & \hat{d}_{22} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\tag{2*}$$

with

$$\begin{aligned}u_1 &= +(\lambda_1 + \lambda_2)y_3 \\ u_2 &= \dot{\hat{\phi}}_{21}v_{21} + \dot{\hat{\phi}}_{22}v_{22} + \dot{\hat{\psi}}_{21}s_{21} + \dot{\hat{\psi}}_{22}s_{22} \\ &= -\left(\frac{1}{\delta_{21}}v_{21}^2 + \frac{1}{\delta_{22}}v_{22}^2 + \frac{1}{\gamma_{21}}s_{21}^2 + \frac{1}{\gamma_{22}}s_{22}^2\right)e_1\end{aligned}\tag{14*}$$

and

$$\dot{v}_{21} + \lambda_2 v_{21} = y_1\tag{11*}$$

$$\dot{v}_{22} + \lambda_2 v_{22} = y_2$$

$$\dot{s}_{21} + \lambda_2 s_{21} = r_1\tag{12*}$$

$$\dot{s}_{22} + \lambda_2 s_{22} = r_2$$

The observer eigenvalues, $-\lambda_1$ and $-\lambda_2$, are arbitrary but distinct negative numbers.

The state \bar{x} of system (1*) may be reconstructed by the equation

$$\bar{x} = \hat{T}(G,D) \begin{bmatrix} \bar{y} \\ \xi \end{bmatrix}$$

where

$$T = \lim_{t \rightarrow \infty} \hat{T}(G,D) = \lim_{t \rightarrow \infty} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ g_{22}(t) + \lambda_2^3 & -\lambda_2^2 & \lambda_2 & 1 & 0 \end{bmatrix}$$

9. Computer Simulation

The system of Section 8 was simulated on a digital computer. The system parameters assumed unknown were b_2 , a_3 , a_{21} , and a_0 . The following values were chosen for simulation:

$$\begin{array}{lll} a_0 = 15 & b_1 = 1 & \gamma_{21} = 1/10 \\ a_1 = 33.5 & b_2 = 2 & \delta_{22} = 1/250 \\ a_2 = 26.0 & \lambda_1 = 10 & \phi_{21}(0) = 180.25 \\ a_3 = 8.5 & \lambda_2 = 5 & \psi_{22}(0) = -100 \end{array}$$

The inputs r_1 and r_2 were chosen as sine waves with frequencies of 3.5 and 5 rad/sec. respectively. The behavior of the two adaptive parameters ϕ_{21} and ψ_{22} are shown in Figure 2 and the (transformed) observer error e_2 is shown in Figure 3.

Conclusions

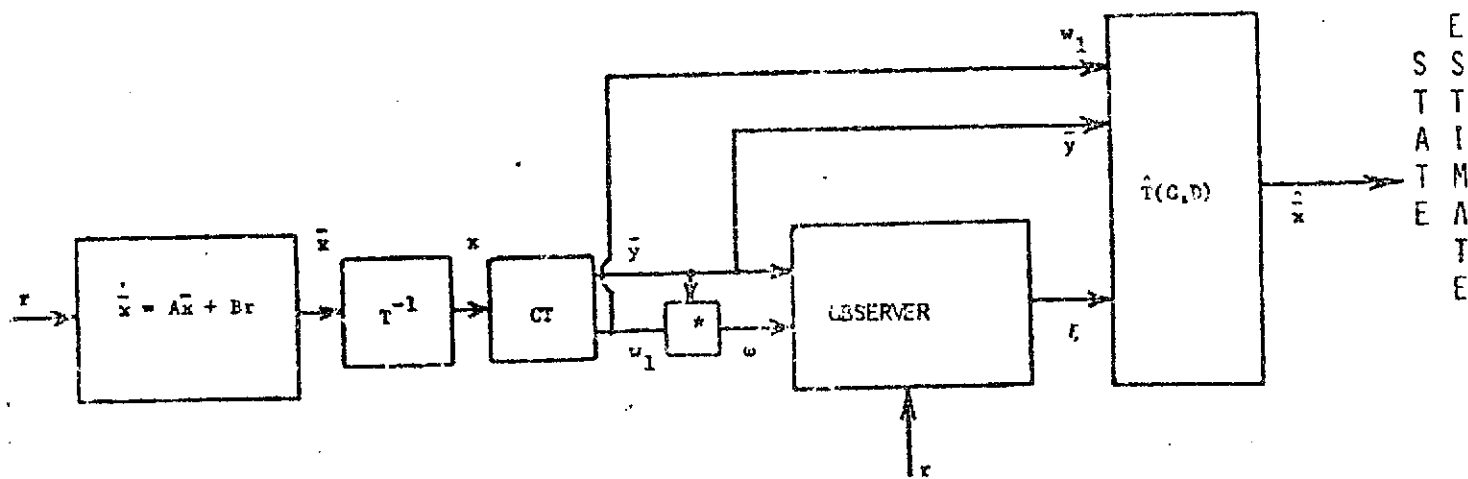
A reduced adaptive observer has been shown to estimate the state of an unknown multivariable system. Significant reduction in the order of the observer and the number of adaptive gains may be obtained by this method. In addition

to generating the state of a system with unknown parameters, partial identification of the parameter is accomplished. Full freedom is allowed in the selection of observer eigenvalues, thus allowing some suppression of inherent system noise.

At present no other reduced adaptive observer has been reported in the literature.

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* Output Transformation (c.f. Section 6)

FIGURE 1 ADAPTIVE OBSERVER SCHEME

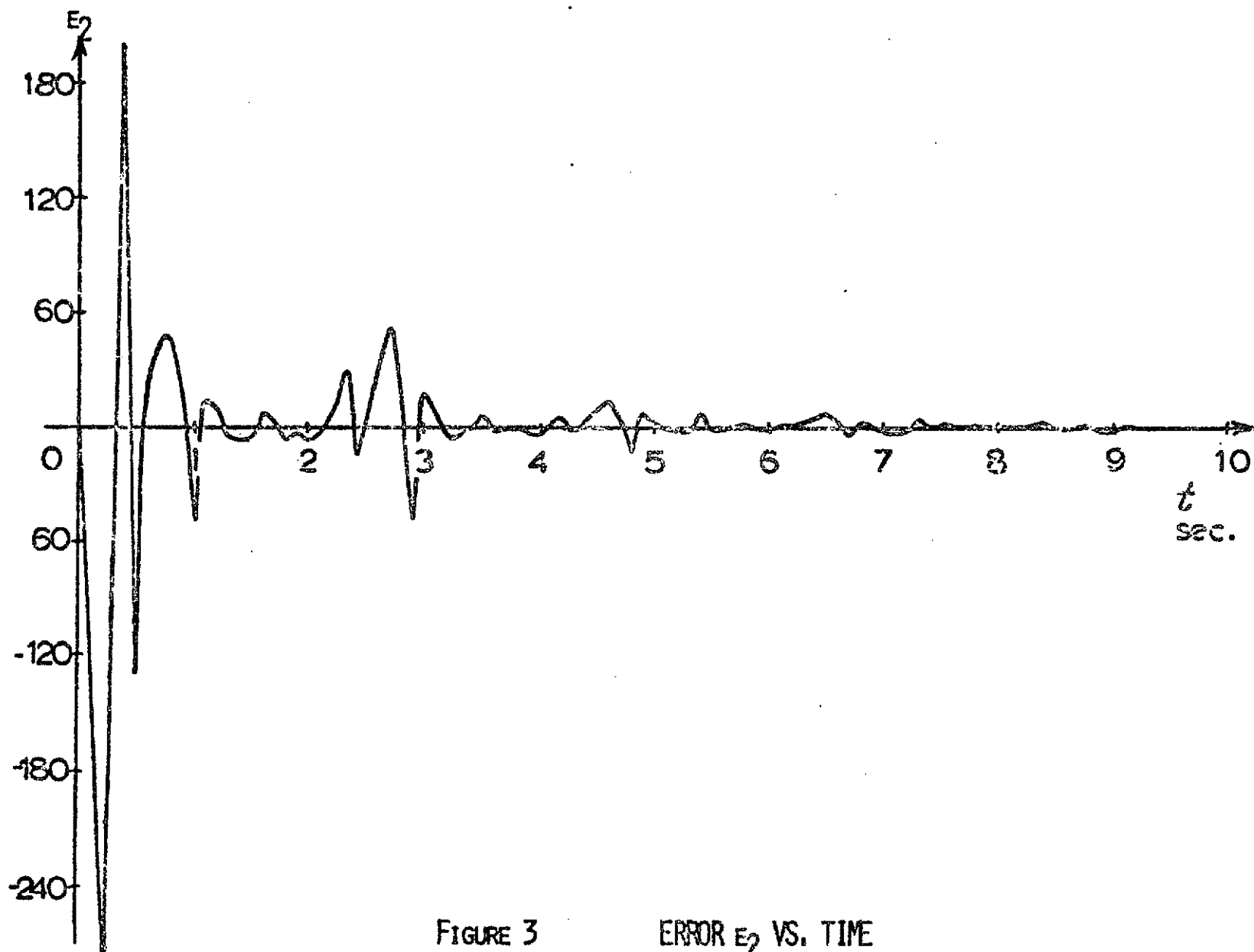


FIGURE 3 ERROR E_2 VS. TIME